Capacity-Constrained Point Distributions: A Variant of Lloyd's Method

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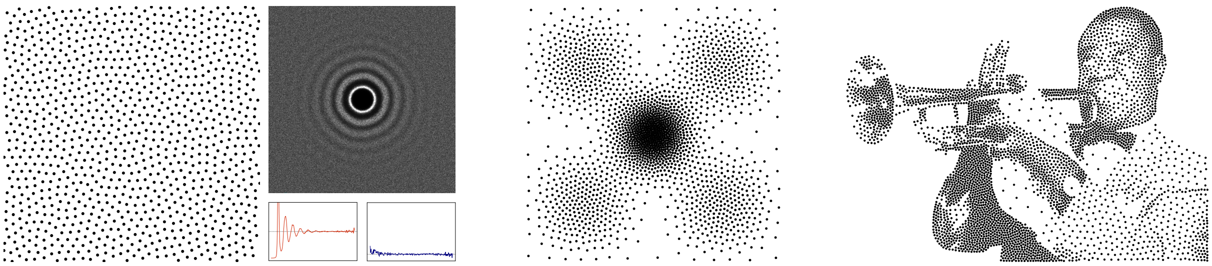


Figure 1: *(Left) 1024 points with constant density in a toroidal square and its spectral analysis to the right; (Center) 2048 points with the density function* $\rho = e^{(-20x^2-20y^2)} + 0.2\sin^2(\pi x)\sin^2(\pi y)$*; (Right) 4096 points with a density function extracted from a grayscale image.*

Abstract

We present a new general-purpose method for optimizing existing point sets. The resulting distributions possess high-quality blue noise characteristics and adapt precisely to given density functions. Our method is similar to the commonly used Lloyd's method while avoiding its drawbacks. We achieve our results by utilizing the concept of capacity, which for each point is determined by the area of its Voronoi region weighted with an underlying density function. We demand that each point has the same capacity. In combination with a dedicated optimization algorithm, this capacity constraint enforces that each point obtains equal importance in the distribution. Our method can be used as a drop-in replacement for Lloyd's method, and combines enhancement of blue noise characteristics and density function adaptation in one operation.

CR Categories: I.3.3 [Computer Graphics]: Picture/Image Generation—Antialiasing I.4.1 [Image Processing and Computer Vision]: Digitization and Image Capture—Sampling

Keywords: importance sampling, Lloyd's method, blue noise, capacity constraint, Voronoi tessellations, Poisson disk point sets

1 Introduction

Point distributions are ubiquitous in computer graphics and are used in such diverse domains as sampling, object and primitive distribution, halftoning, point-based modeling and rendering, and geometry processing [Pharr and Humphreys 2004]. One desirable property of point distributions in these contexts is that they possess blue noise characteristics, with large mutual distances between points and no apparent regularity artifacts. Another desirable property is that point distributions adapt to a given density function in the sense that the number of points in an area is proportional to the density.

The iterative method by Lloyd [1982] is a powerful and flexible technique that is commonly used to enhance the spectral properties of existing distributions of points or similar entities. However, the results from Lloyd's method are satisfactory only to a limited extent. First, if the method is not stopped at a suitable iteration step, the resulting point distributions will develop regularity artifacts, as shown in Figure 2. A reliable universal termination criterion to prevent this behavior is unknown. Second, the adaptation to given heterogenous density functions is suboptimal, requiring additional application-dependent optimizations to improve the results.

We present a variant of Lloyd's method which reliably converges towards distributions that exhibit no regularity artifacts and precisely adapt to given density functions. Like Lloyd's method it can be used to optimize arbitrary input point sets to increase their spectral properties while avoiding its drawbacks. We achieve the quality of our results by applying a so-called *capacity constraint*. This constraint enforces that each point in a distribution has the same capacity. Intuitively, the capacity can be understood as the area of the point's corresponding Voronoi region weighted with the given density function. By demanding that each point's capacity is the same, we ensure that each point obtains equal importance in the resulting distribution. This is a direct approach to generating uniform distributions, whereas Lloyd's method achieves such distributions only indirectly by relocating the sites into the corresponding centroids.

Based on this capacity constraint, we utilize an iterative algorithm based on [Balzer and Heck 2008] to optimize given point distributions. An evaluation of the results confirms that our constraint is responsible for improved blue noise characteristics and precise adaptation to density functions. Due to its similarity to Lloyd's method, our method can be used as a substitute in applications that currently benefit from Lloyd's method, albeit at a higher computational cost. In addition, it combines enhancement of blue noise characteristics and density function adaptation in a single operation.

The remainder of the paper is structured as follows: In the next section we review work related to Lloyd's method and its application in computer graphics. In Section 3 we discuss the theoretical background of Lloyd's method and present our variant of capacity-constrained point distributions. In Section 4 we provide a critical evaluation of our results and those of Lloyd's method. Finally, in Section 5 we draw some conclusions.

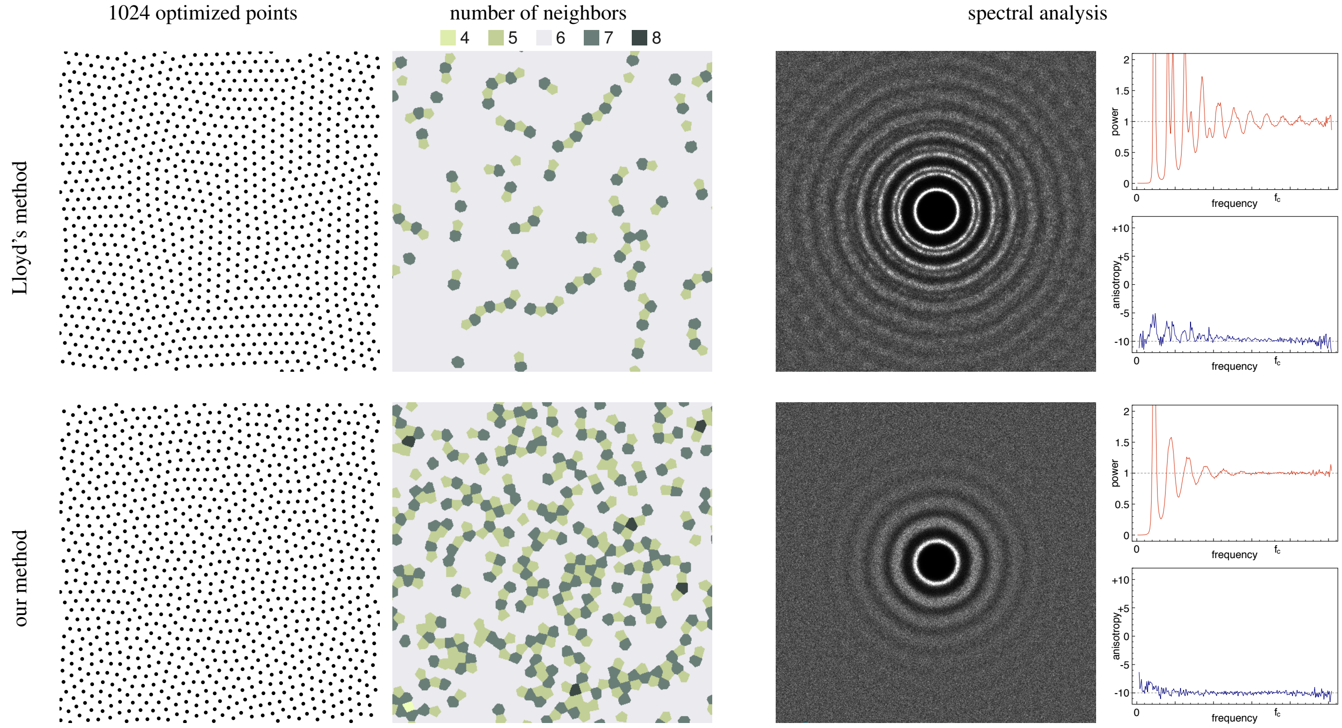


Figure 2: *Lloyd's method generates point distributions with regular structures if it is not stopped manually. This becomes evident by color coding the number of neighbors for the Voronoi region of each point. The example set of 1024 points was computed with Lloyd's method to full convergence and contains large patches of hexagons. These patches are separated by just a few heptagons and pentagons in between. Due to the higher variance in the number of neighbors, the result of our method does not exhibit such regularities. The spectral analysis to the right confirms this. The mean periodogram is much smoother, with less turbulent radial power and lower anisotropy.*

2 Related Work

To solve the aliasing problem in computer graphics, Dippé and Wold [1985], Cook [1986], and Mitchell [1987] introduced non-uniform sampling. In this process, Poisson disk distributions were identified as a spectrally near optimal sampling pattern, having blue noise characteristics similar to those of receptors in primate's retina [Yellott 1983]. The associated early dart throwing algorithm to generate such point distributions was provided by Cook [1986] and accelerated by McCool and Fiume [1992] with their relaxation dart throwing algorithm. McCool and Fiume [1992] also introduced Lloyd's method to computer graphics, as they recognized that it further improved the spectral properties of their results. They already found that Lloyd's method should be stopped after a few iterations to prevent regularity artifacts.

Since then, several techniques have emerged that try to generate point distributions with blue noise characteristics while reducing the runtime. These techniques typically involve some tiling approach, allow progressive refinement, and decrease the generation time to real-time speeds. The approaches are either deterministic tilings, such as the Penrose or polyomino tilings of Ostromoukhov et al. [2004; 2007], or they work on precomputed sets of single tiles, such as [Hiller et al. 2001; Lagae and Dutré 2006; Kopf et al. 2006]. Each of these techniques requires preprocessing to gain blue noise characteristics, and incorporates Lloyd's method at that point.

In addition to these complex construction methods, recent techniques showed significant progress in the generation of Poisson disk distributions, equivalent to those generated by dart throwing, such as [Jones 2006; Dunbar and Humphreys 2006; White et al. 2007; Wei 2008]. Since none of these techniques is able to combine blue noise characteristics, progressive refinement, and density function adaptation in an equally effective way, Lloyd's method remains pivotal in the context of sampling in computer graphics.

Lloyd's method was not used for the adaptation to given density functions in the aforementioned real-time techniques. However, it was used for density adaptation by offline techniques, such as [Secord 2002] for non-photorealistic rendering, [Kollig and Keller 2003] for image based lighting using high dynamic range images, or [Surazhsky et al. 2003] for geometry processing. In these cases, Lloyd's method had to be combined with other heuristics or optimization approaches to improve the adaptation to density functions. In contrast, Chen [2004] suggested a mesh optimization method based on centroidal Delaunay triangulations, which tends to produce regularity artifacts as well.

3 Capacity-Constrained Point Distributions

In this section, we present our method for generating capacity-constrained point distributions. First, we provide theoretical background by introducing centroidal Voronoi tessellations and presenting Lloyd's algorithm as a construction method for such tessellations. Then we introduce and motivate the concept of capacity and describe an algorithm for computing capacity-constrained Voronoi tessellations. Finally, we present our method for generating capacity-constrained point distributions.

Throughout this section and the following Section 4, we refer to the points of our distributions as *sites*. In contrast, the term *point* is used for the points of the underlying space.

Centroidal Voronoi Tessellations A set S of n sites in Euclidean d-space $\mathbb{R}^d$ induces a partition of $\mathbb{R}^d$ into n regions. Each such region V_i belonging to a site $s_i \in S$ consists of all points x that are closer to s_i than to any other site $s_j \in S$, $i \neq j$. This partition is known as the *Voronoi tessellation* $\mathcal{V}(S)$ of S in $\mathbb{R}^d$. A *centroidal Voronoi tessellation* is a Voronoi tessellation in a bounded space $\Omega \subset \mathbb{R}^d$ with the property that each site s_i coincides with the centroid of its Voronoi region V_i. The *centroid* p_i of a Voronoi region V_i is calculated as

$$p_i = \frac{\int_{V_i} x\rho(x)dx}{\int_{V_i} \rho(x)dx}, \tag{1}$$

where $\rho(x) \geq 0$ is a given density function defined on Ω. In other words, a centroid is the center of mass of a Voronoi region with respect to the density function. The importance of centroidal Voronoi tessellations is established by their relation to the energy function

$$\mathcal{F}(S, \mathcal{V}) = \sum_{i=1}^{n} \int_{V_i} \rho(x) \left| x - s_i \right|^2 dx, \tag{2}$$

where $V_i \in \mathcal{V}$ and $s_i \in S$. A necessary condition for $\mathcal{F}$ to be minimized is that $\mathcal{V}$ is a centroidal Voronoi tessellation of S. For a comprehensive treatment of the topic, we refer to [Du et al. 1999; Okabe et al. 2000].

Lloyd's Method A common way to generate centroidal Voronoi tessellations is the method introduced by Lloyd [1982]. This iterative algorithm performs the following steps:

1. generate the Voronoi tessellation $\mathcal{V}(S)$ in Ω;
2. move each site $s_i \in S$ to the centroid p_i of the corresponding Voronoi region $V_i \in \mathcal{V}$;
3. if the new sites in S meet some convergence criterion, then terminate; otherwise return to step 1.

Due to the fact that each relocation of a site to its centroid reduces the energy $\mathcal{F}$, the algorithm converges to a local minimum of $\mathcal{F}$, in which each site coincides with the centroid of its Voronoi region. Hence, Lloyd's method is a gradient descent method that minimizes the energy function $\mathcal{F}$ [Du et al. 1999]. Similar gradient descent methods for $\mathcal{F}$, such as [Liu et al. 2008], generate equivalent results.

Lloyd's method can either be performed in continuous or in discrete spaces. An implementation in continuous spaces is usually faster, but also more complex and less flexible. Especially the incorporation of density functions is intricate in the continuous case. It is therefore common to use discrete implementations, where the bounded space Ω with density function ρ is represented by a set X of m sample points. Based on this point set X, the Voronoi tessellation is formed by an assignment $A : X \to S$, which assigns each point in X to the closest site in S. Consequently, the energy function $\mathcal{F}$ in Equation 2, which is minimized by Lloyd's method, is substituted by

$$\mathcal{F}'(X, A) = \sum_{i=1}^{m} \left| x_i - A(x_i) \right|^2. \tag{3}$$

The computational time complexity of Lloyd's method in two-dimensional continuous space with constant density is $O(n \log n)$ for each iteration [Du and Emelianenko 2006], whereas for discrete space with m sample points it is $O(m \log n)$. The memory complexity of Lloyd's method is at least $O(n)$, or $O(n + m)$ if it is based on a non-regular discrete point set.

Capacity Constraint The concept of capacity is defined as follows. Consider a set S of n sites that determines a Voronoi tessellation $\mathcal{V}(S)$ in the bounded space Ω with the density function $\rho(x)$. The *capacity* $c(s_i)$ of a site $s_i \in S$ with respect to its Voronoi region $V_i \in \mathcal{V}$ is defined as

$$c(s_i) = \int_{V_i} \rho(x)dx. \tag{4}$$

We say that a distribution of sites in S adapts optimally to the density function ρ, if the capacity of each site $s_i \in S$ fulfills the constraint

$$c(s_i) = c^*, \tag{5}$$

where c^* is a globally defined scalar value given by

$$c^* = \frac{\int_\Omega \rho(x)dx}{n}. \tag{6}$$

The term $c(s_i) = c^*$ for each site $s_i \in S$ is our capacity constraint.

Intuitively, the capacity of a site is equivalent to the area of its corresponding Voronoi region weighted with the density function. Hence, our capacity constraint enforces that each site in a distribution is equally important. This is directly related to the approach of importance sampling [Halton 1970].

Capacity-Constrained Voronoi Tessellations An arbitrary distribution of n sites in S usually does not fulfill the capacity constraint $c(s_i) = c^*$ for all sites $s_i \in S$. Rather, such a distribution has to be determined by generating a *capacity-constrained Voronoi tessellation* $\mathcal{V}(S, C)$ based on a set C of n arbitrary non-negative capacity values. In our case, all elements of C have the same value c^*.

In general, the Voronoi tessellation of a fixed set S of sites will not meet a given capacity constraint C. Aurenhammer et al. [1998] showed, however, that any set of sites will always have a power tessellation which does meet the capacity constraint. They also showed that a power tessellation in discrete space minimizes the function $\mathcal{F}'$ in Equation 3, regardless of the capacity constraint C. Note that the power tessellation is a generalization of the ordinary Voronoi tessellation.

Based on this work of Aurenhammer et al. [1998], we presented an approach for the computation of capacity-constrained Voronoi tessellations $\mathcal{V}(S, C)$ in n-dimensional discrete spaces [Balzer and Heck 2008]. In this approach we start with a random assignment $A : X \to S$ of m points in X to n sites in S that fulfills the capacity constraint C. In this discrete case, the capacity constraint C is fulfilled if the number of points that are assigned to each site $s_i \in S$ equals the capacity $c(s_i) \in C$. After this initialization, we minimize $\mathcal{F}'(X, A)$ by continually swapping the assignment between two points in X that belong to different sites in S if and only if such swap reduces the energy in $\mathcal{F}'$. The restriction to swap operations guarantees that the capacity constraint C is maintained throughout the minimization. The assignment swaps are performed until no further swap reduces the energy and the algorithm stops in a stable state that represents a local minimum of $\mathcal{F}'$. Such local minimum is achieved by optimizing the assignment A, not by relocating the sites in S as it is done by Lloyd's method. Our approach is further formalized in Algorithm 1.

The result of our algorithm is an assignment $A : X \to S$ that represents a power tessellation of S in X and fulfills the capacity constraint C. The computational time complexity of the algorithm is $O(n^2 + nm \log \frac{m}{n})$, and its memory complexity is $O(n + m)$. For more details consult our paper [Balzer and Heck 2008].

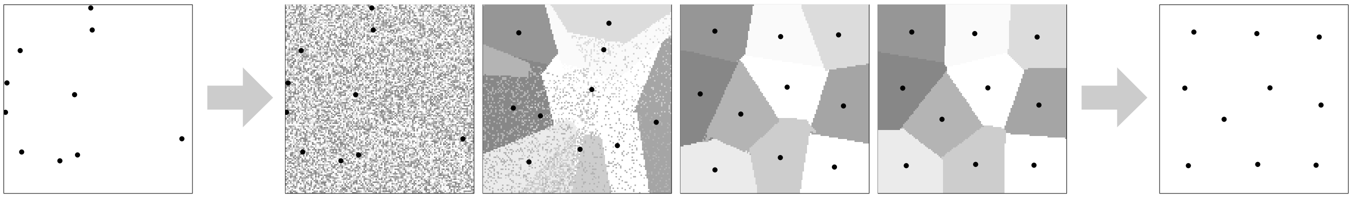


Figure 3: *Our method takes an existing site distribution and transfers it to a random discrete assignment in which each site has the same capacity. This assignment is then optimized so that Voronoi regions are formed and sites are relocated to the centroids of their regions, while simultaneously maintaining the capacity for each site. The optimization stops at an equilibrium state with the final site distribution.*

Algorithm 1: Capacity-Constrained Voronoi Tessellation

Input: Set S of n sites, Set X of m points, Set C of n capacities with $\sum C = m$
Output: Power tessellation $A : X \rightarrow S$ that fulfills the capacity constraint C
Initialize a random assignment $A : X \rightarrow S$ that fulfills the capacity constraint C;
repeat
  stable := *true*;
  foreach pair of sites (s_i, s_j) with $s_i, s_j \in S, i < j$ **do**
    Initialize two heap data structures H_i, H_j;
    foreach point x_i with $A(x_i) = s_i$ **do**
      insert x_i into H_i with key $|x_i - s_i|^2 - |x_i - s_j|^2$;
    foreach point x_j with $A(x_j) = s_j$ **do**
      insert x_j into H_j with key $|x_j - s_j|^2 - |x_j - s_i|^2$;
    while $|H_i| > 0$ and $|H_j| > 0$ and $max(H_i) + max(H_j) > 0$ **do**
      modify the assignment A to $A(max(H_i)) := s_j$ and $A(max(H_j)) := s_i$;
      remove $max(H_i)$ from H_i and $max(H_j)$ from H_j;
      stable := *false*;
until *stable* = *true* ;

Capacity-Constrained Method The capacity-constrained assignments generated with the foregoing algorithm represent tessellations that differ from ordinary Voronoi tessellations because they are generated under the capacity constraint without modifying the site locations. This difference is almost eliminated if two additional constraints are fulfilled: first, the capacity for each site must be equal, and second, each site must reside in the centroid of its Voronoi region. Our experiments showed that in this case less than 3 percent of all points in X are assigned to a site $s_i \in S$ in the capacity-constrained Voronoi tessellation $\mathcal{V}(S, C)$ while they are assigned to another site $s_j \in S$ in the ordinary Voronoi tessellation $\mathcal{V}(S)$ of the same set of sites S. Fortunately, the first constraint is already assured by our capacity constraint $c(s_i) = c^*$ for each site, and the second constraint can be easily achieved by iteratively moving the sites in the centroids of their regions, similar to the approach by Lloyd.

To generate our capacity-constrained distributions we perform the following steps, which are illustrated by Figure 3:

0. create a set X of m points that is a discrete representation of the space Ω weighted with the density function ρ;

1. generate the capacity-constrained Voronoi tessellation $\mathcal{V}(S, C)$ for the set S of n sites as an assignment $A : X \rightarrow S$ where each site $s_i \in S$ has the capacity $c(s_i) = \frac{m}{n}$;

2. move each site $s_i \in S$ to the center of mass of all points $x_i \in X$ that are assigned to this site, $A(x_i) = s_i$;

3. if the new sites in S meet some convergence criterion, then terminate; otherwise return to step 1.

These steps are a variant of Lloyd's method, in which the generation of the Voronoi tessellation is substituted with the capacity-constrained optimization in discrete space.

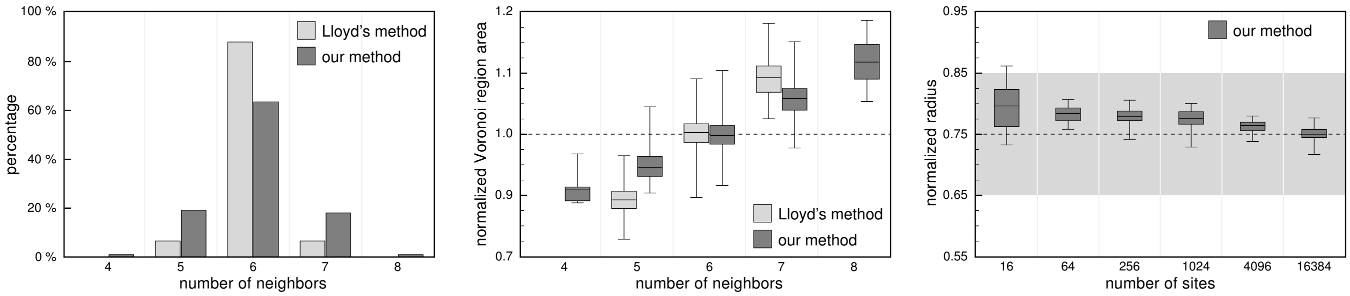
The computational time complexity for each iteration of our method for n sites and m points is $O(n^2 + nm \log \frac{m}{n})$, which is higher than Lloyd's method with $O(m \log n)$ for each iteration. Point optimizations are often performed as a pre-processing step, and in these situations the running time is not a significant drawback. Thus, our method allows the same application scenarios as Lloyd's method, especially since it converges much faster. After only about five iterations, our results exhibit very good properties, with most of the remaining computation time necessary for the subtle improvements towards the final equilibrium state, while Lloyd's method converges much slower and is highly dependent on the initial distribution.

4 Evaluation of Results

In the previous sections, we stated that Lloyd's method generates point distributions that possess suboptimal blue noise characteristics and do not adapt well to given density functions. We also stated that the results of our method are better with respect to these two properties. In this section, we substantiate these claims by an extensive analysis of the results of Lloyd's method and those of our method. Furthermore, we illustrate why our capacity constraint is reasonable. Finally, we present two application examples.

For the evaluation of blue noise characteristics, we restrict ourselves to point sets in a toroidal square with constant density. This is necessary to evaluate spectral properties, but the findings are still valid for other spaces and arbitrary density functions.

Blue Noise Characteristics Figure 2 shows a representative distribution of 1024 random points that was generated with Lloyd's method until no further relocation of sites occurred. The distribution clearly exhibits patches of hexagonal lattices. These regular structures are reflected in the corresponding mean periodogram [Ulichney 1987], which is not smooth, having turbulent radial power and significant anisotropy. This mean periodogram is calculated from the results of Lloyd's method for 10 different initial random sets of 1024 points. In contrast, our capacity-constrained distributions, which were computed with the same initial point sets, show substantially better results: regularities are much less apparent in the representative example, and the mean periodogram is smoother, with less turbulent radial power and lower anisotropy. This observation is independent of the initial point distribution.



(a) number of neighbors percentage (b) normalized Voronoi region area (c) normalized radius α

Figure 4: Comparison of result sets with 1024 sites: *(a) our method generates less hexagons than Lloyd's method. (b) our method generates regions with more uniform areas than Lloyd's method.* Comparison of result sets with different numbers of sites in (c): *our results have a normalized radius α within the preferable interval $[0.65, 0.85]$, and near the reference value $\alpha \approx 0.75$ by Lagae and Dutré [2008].*

The reason for the development of regularity artifacts when using Lloyd's method is the minimization of the energy function $\mathcal{F}$ in Equation 2. The minimal value for this function in two-dimensional space would result if all Voronoi region were circles of equal size, which is impossible. Instead, a global minimum of $\mathcal{F}$ is achieved if the sites form a regular hexagonal lattice. Since such lattice cannot be generated for any number of sites, tessellations approach local optimality with respect to $\mathcal{F}$ by embedding non-hexagons in between. This is in line with observations in the domain of circle packing [Szabó et al. 2007]. Furthermore, a region that forms a n-gon is less optimal with respect to $\mathcal{F}$ than a region that forms a $(n+1)$-gon having the same area. Therefore, an arbitrary distribution of sites optimized by Lloyd's method approaches a tessellation which consists of hexagonal patches that are as large as possible. These patches are connected by a few smaller n-gons with $n < 6$, and a few larger n-gons with $n > 6$. Our constraint of equal capacity significantly improves on this behavior due to the fact that all regions are equally sized throughout the minimization, so that the energy can no longer benefit from area differences between n-gons and $(n+1)$-gons. This effect is illustrated by the neighbor diagrams next to the point sets in Figure 2. These diagrams show hexagonal patches for Lloyd's method, whereas our result is much more heterogeneous with fewer noticeable regularities.

These theoretical considerations are confirmed by our experiments, as illustrated in Figures 4(a) and 4(b). Here, we present an analysis of 10 different distributions of 1024 sites that have been optimized with Lloyd's method until all sites were stable. This analysis identifies that 87.8% of the Voronoi regions are hexagons, 6.1% are pentagons, and again 6.1% are heptagons. Furthermore, the pentagons are significantly smaller than the hexagons, which in turn are significantly smaller than the heptagons. A similar analysis of the results of our method based on the same initial site sets shows that our method yields a larger variance in shape, with only 69.6% hexagons, 15.5% pentagons, 14.7% heptagons, and even a small fraction of quadrilaterals and octagons. Furthermore, the area of these groups of n-gons are more uniform.

Termination Criterion for Lloyd's Method The development of regularity artifacts in distributions generated with Lloyd's method is well known in computer graphics. A common solution is to stop Lloyd's method after a few iterations. Lagae and Dutré [2008] suggest the normalized Poisson disk radius $\alpha \in [0, 1]$ as a quality measure for point distributions. This radius α is equal to zero if any two points in the distribution coincide, equal to one for a hexagonal lattice, and is considered optimal for $\alpha \approx 0.75$, the value employed by Lagae and Dutré [2008] for a reference point set obtained via dart throwing. The convergence of α can be utilized as a termination criterion, where we stop Lloyd's method as soon as α is stable. However, our experiments show that this approach is rather unreliable. Figure 5 demonstrates this for a set of 1024 points as well as a set of $512^2 = 262144$ points that is used for the sampling of the zone plate test function $(x, y) \mapsto \sin(x^2 + y^2)$. The example shows that α strongly depends on the number of points and the initial distribution. This is explained by the character of the Poisson disk radius, which is determined by the smallest distance between any two points, while being a representative for the overall distribution. Using Lloyd's method, it is common that a small fraction of closely packed points remains stable, while the overall distribution is still changing.

Identifying a universal termination criterion for Lloyd's method with respect to the blue noise characteristics is an unsolved problem. This necessitates either a manual intervention, or an intricate search for an application-specific criterion. In contrast, our method does not approach any critical states and terminates reliably at an equilibrium with good properties. Our results demonstrate better blue noise characteristics for the set of 1024 points with $\alpha \approx 0.77$, as well as a low-artifact sampling of the zone plate test function with $\alpha \approx 0.63$. In general, our method generates results that are close to $\alpha \approx 0.75$, as illustrated by Figure 4(c). Here, we used a large number of different initial point sets generated by different methods such as random sampling or dart throwing as input for our method. The plot shows that all results are within the recommended interval, and close to the reference value $\alpha \approx 0.75$.

Density Function Adaptation In Section 3, we introduced our concept of equal capacity which is directly related to importance sampling in computer graphics. The capacity offers the possibility to measure the quality of a density function adaptation by a distribution of sites via the normalized capacity error

$$\delta_c = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{c(s_i)}{c^*} - 1 \right)^2 . \tag{7}$$

Site distributions that are well adapted to their underlying density function have δ_c close to zero.

We observed in our experiments that if the density is constant, Lloyd's method generates a uniform distribution with a small capacity error. In contrast, if the density is not constant, Lloyd's method generates site distributions with a large capacity error and suboptimal density function adaptation. In particular, areas with high density contain too few sites, and areas with low density contain too many sites. This indicates that Lloyd's method implic-

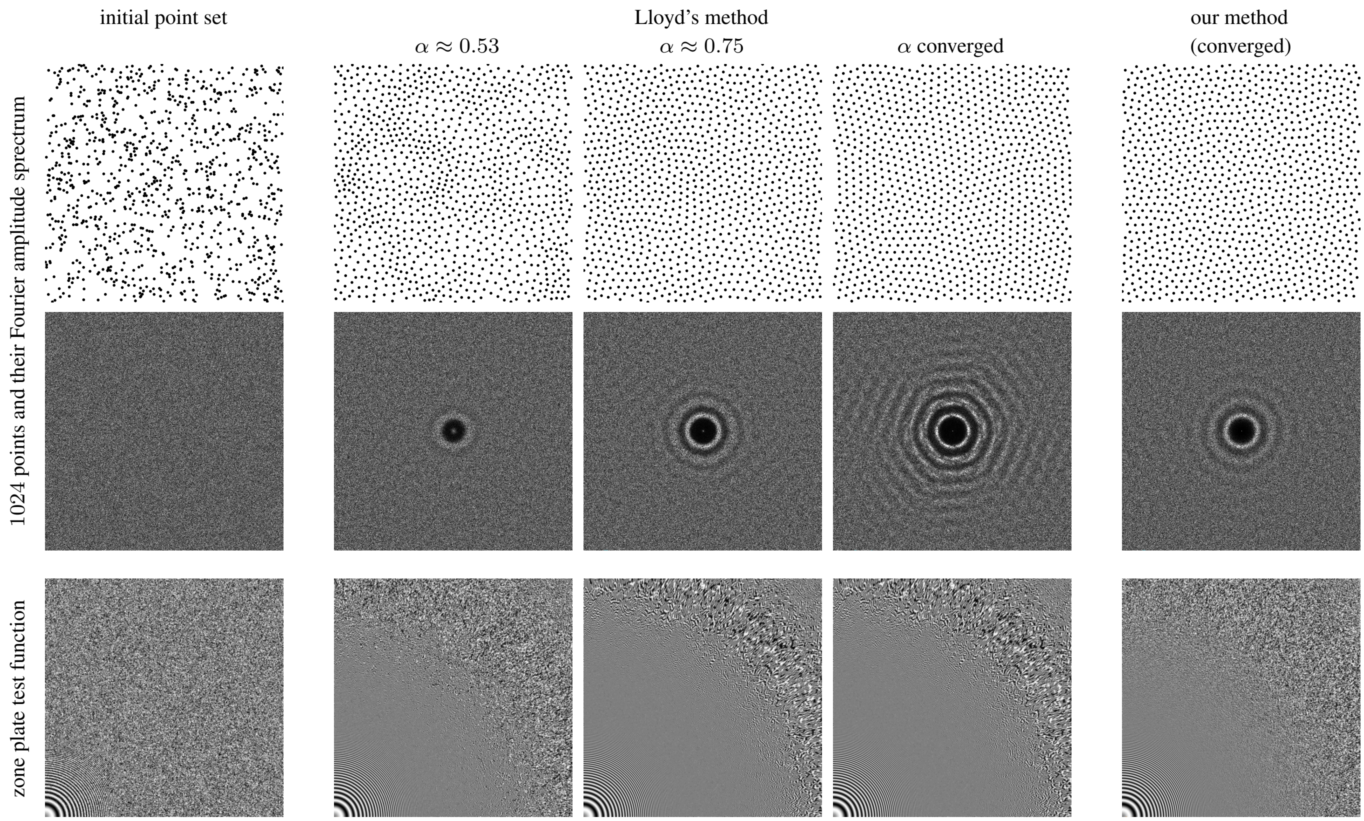


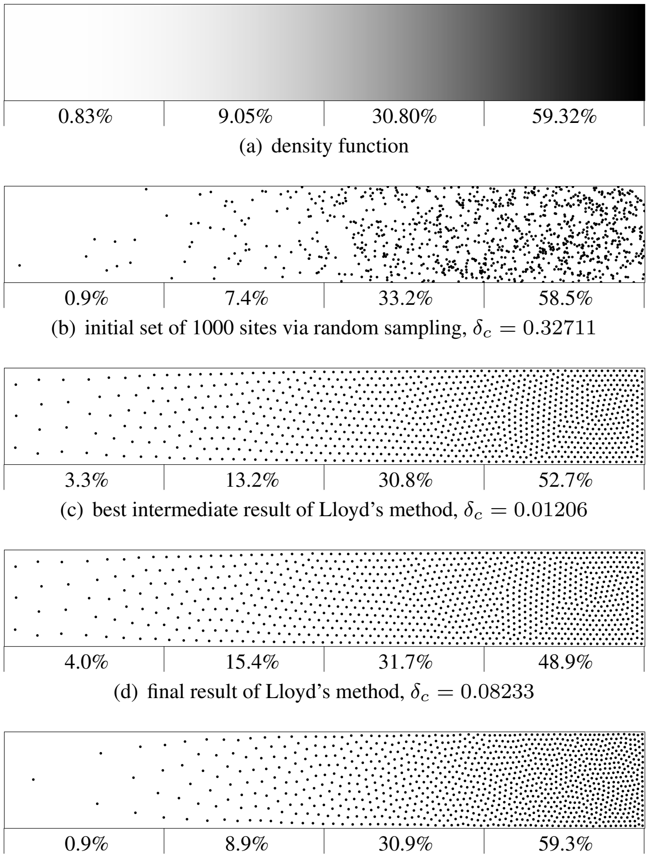
Figure 5: *An initial set of 1024 points is optimized by Lloyd's method. After 40 iterations the points are well distributed with a normalized radius of $\alpha \approx 0.75$ and good blue noise characteristics. Further optimization deteriorates the spectral properties and introduces hexagonal structures. In contrast, $\alpha \approx 0.75$ proves to be ill-suited for the sampling of the zone plate test function with 512^2 points as strong artifacts become apparent. Relying on the convergence of α is also not an option as only marginally fewer artifacts can be observed. In this sampling scenario, stopping Lloyd's method after about 10 iterations with $\alpha \approx 0.53$ would provide the best sampling results. Our method converges reliably to an equilibrium with better properties in both scenarios.*

itly blurs the density function. This erroneous behavior can be attributed to the local operation of centroid relocation, which disregards any global characteristics of the underlying density function.

The implicit blurring of the density function in Lloyd's method is illustrated by Figure 6. Here, a quadratic ramp is used as density function in (a). The percentages indicate the density ratios that are contained in each quarter of the ramp. An initial set of 1000 sites in (b) is chosen via random sampling, and has a capacity error of $\delta_c = 0.25622$. The percentages of the quarters denote the ratios of contained sites, showing a reasonable first approximation of the density function. By applying Lloyd's method to this initial distribution, at first the capacity error decreases to a minimum of $\delta_c = 0.01206$ in (c). Afterwards, the capacity error again steadily increases. The final result in (d) has a capacity error of $\delta_c = 0.08233$, where the leftmost quarter contains only 0.83% of the overall density but 4.0% of the sites, and the rightmost quarter contains 59.32% of the overall density but only 48.9% of the sites. This behavior illustrates the suboptimal density adaptation of Lloyd's method. Of course, the capacity error can be used as a termination criterion for Lloyd's method, but even the minimum capacity error is in general far from optimal. Usually, such a minimum already contains regularity artifacts, which can be observed in image (c) as well. In contrast, our method generates the distribution in (e) from the same initial sites. It adapts precisely to the given density, and yields a capacity error of $\delta_c = 0.00131$. The ratios of sites and density per quarter are also highly correlated.

Application Examples To further demonstrate the quality of our distributions beside their spectral properties, we present two application examples. Non-photorealistic stippling places small black dots in such a way that their distribution gives the impression of tone. One prominent example for this technique is the approach by Secord [2002], which uses weighted centroidal Voronoi tessellations to generate stipple drawings from a given grayscale image. A result of this approach is shown in Figure 7. By applying our capacity-constrained method without any modification to the same grayscale image, we obtain the stipple drawing in (c), with a computation time of 17 minutes on Intel Core 2 hardware. Both results use 20000 dots with equal radius. Our result reproduces the contrast of the grayscale original more faithfully and contains fewer regularity artifacts.

The second application samples high dynamic range images with respect to their radiance and/or luminance. The resulting samples are then used for image based lighting. The density distribution in such images is characterized by large regions with low density, and small light emitting regions with density values that are orders of magnitude higher. Although Lloyd's method usually fails in the density adaptation of such images, it is nevertheless employed in combination with improvements that reduce its erroneous behavior [Kollig and Keller 2003; Ostromoukhov et al. 2004]. By applying our capacity-constrained method without any modification, we are able to create a distribution that precisely adapts to the underlying lighting information. An example of a resulting importance sam-



(a) density function

(b) initial set of 1000 sites via random sampling, $\delta_c = 0.32711$

(c) best intermediate result of Lloyd's method, $\delta_c = 0.01206$

(d) final result of Lloyd's method, $\delta_c = 0.08233$

(e) final result of our method, $\delta_c = 0.00131$

Figure 6: *The quadratic ramp in (a) is used as density function. Starting with the initial sites in (b), Lloyd's method continually blurs the density function. In contrast, our result shows precise adaptation. The percentages indicate the amount of density or the number of points contained in each quarter.*

pling of the luminance of a high dynamic range image is given in Figure 8. Here, the distribution of 3000 samples exhibits no regularity artifacts and extracts even subtle features in areas with maximum density such as the cross of the main window. The computation time was 8 minutes on Intel Core 2 hardware.

5 Conclusion

We presented a new general purpose method for optimizing point distributions. Our method improves on the established Lloyd's method by utilizing the concept of capacity as an optimization constraint. The resulting capacity-constrained point distributions exhibit improved blue noise characteristics, having a large mutual distance and no apparent regularity artifacts. Arbitrary density functions can be utilized to control our point distributions and, again, the capacity constraint guarantees their precise adaptation.

An important advantage of our method is the reliable quality of the results. After only a few iterations, the point distributions possess significantly improved blue noise characteristics and are precisely distributed according to the underlying density function. In the remaining process, these properties are further improved, until an equilibrium state is finally reached. The quality of the final result is neither dependent on the initial distribution nor is it worse than earlier results during the optimization.

Our method is sufficiently simple, having a similar implementation effort as Lloyd's method. Simultaneously, it avoids the drawbacks of Lloyd's method and does not require manual intervention or an application-dependent termination criterion. In addition, the enhancement of blue noise characteristics and the density adaptation can be combined in one operation. In most applications, our method can be used as a drop-in replacement for Lloyd's method. To support its implementation, we provide C++ code through an open source project at `http://ccvt.googlecode.com`.

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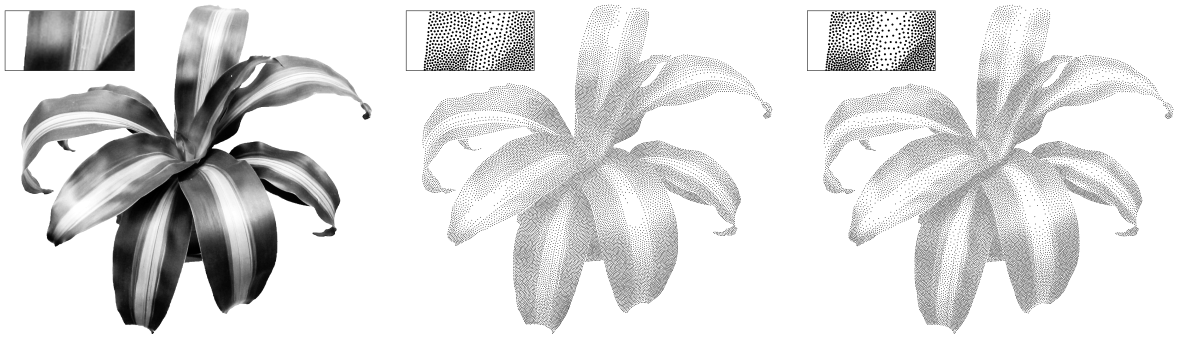

(a) original grayscale image (b) result by Secord [2002] (c) our result

Figure 7: *The grayscale image in (a) is used as density function to generate stipple drawings. The result of our unmodified method in (c) exhibits no regularities and higher local contrast than the reference in (b). Each stipple drawing uses 20000 points with the same radius.*

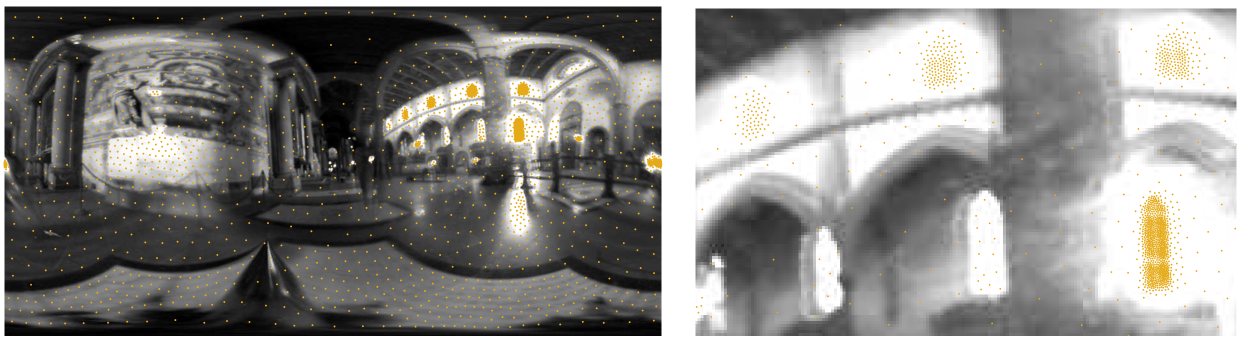

Figure 8: *The left image shows a luminance sampling of the high dynamic range image "Galileo's Tomb" (courtesy of Paul Debevec) using 3000 points. A detail of the same point distribution is shown in the right image. Our method generates no regularity artifacts and extracts even subtle features such as the the cross of the main window.*

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