Tiled Blue Noise Samples

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Abstract

Sampling points with blue noise spectral characteristics are best suited for antialiasing. Compared to Poisson disk sampling, Lloyd's relaxation scheme yields samples that approximate these superior frequency characteristics much better. We complete the elementary algorithm by new techniques that allow to generate hierarchical point sets for adaptive sampling that can be efficiently tiled over the sampling plane in an aperiodic way while preserving the blue noise characteristics.

1 Introduction

When sampling in computer graphics, very often the sampling rate cannot be chosen large enough to fulfill Shannon's sampling theorem. Then a convenient method is to map the low frequency aliasing artifacts to noise using random sampling.

In [12, 8] it is stated that low frequencies have to be avoided in the sampling patterns, otherwise large scale artifacts will become visible. Poisson disk and blue noise distributions [9] are random sampling patterns that avoid such frequencies in an optimal way due to their minimum distance criteria but are expensive to generate.

Since the generation of such sample points takes considerable effort, the idea of tiling sample point sets as already mentioned by Dippé and Wold [3] can be exploited: tiling allows for an exact control of the number of samples for a given subarea of the sampling plane and the expensive generation schemes need to be performed for a small number of master tiles only. However Dippé and Wold did not provide an efficient tiling scheme.

We present a solution to this open problem: based on the elementary relaxation algorithm presented in [8], we introduce a new technique for generating hierarchical random sample point sets with blue noise distribution that allow for adaptive sampling. These tiles of progressive sampling points are constructed so that they allow to tile the sampling plane in an aperiodic way while exposing the favorable blue noise characteristics over the whole plane. Our new scheme only requires a very small fraction of memory for tabulating a small number of hierarchical blue noise sample pattern tiles.

1.1 Previous Work

For an extensive survey on sampling techniques in the context of computer graphics we refer to [5]. We focus on previous work by Mitchell [9] and later by McCool and Fiume [8]. In [10] Mitchell introduced a scanline algorithm for the approximate generation of sample points with blue noise characteristics. While this approach was very limited for adaptive sampling, McCool and Fiume presented a generalization of the dart throwing approximation for creating hierarchical Poisson disk samples.

The dart throwing algorithm mimics the stochastic Poisson disk process by successively adding random points to a points set. A new point is accepted if and only if no other point is inside the disk of specified radius centered at the new point. As this process is not guaranteed to terminate, McCool and Fiume reduce the disk radius slightly during dart throwing. This allows to generate sets of arbitrary size in a fixed region but reduces the spectral quality of the point pattern, since the disk radius of a Poisson disk process is a lower bound for the minimal distance between two sampling points. It is difficult to choose this minimal radius: a small radius allows for very non-uniform distributions while a too large radius inhibits convergence.

In order to overcome the problems of a too small radius McCool and Fiume apply Lloyd's method [7] as a post process. The relaxation technique is applied to the points generated by dart throwing

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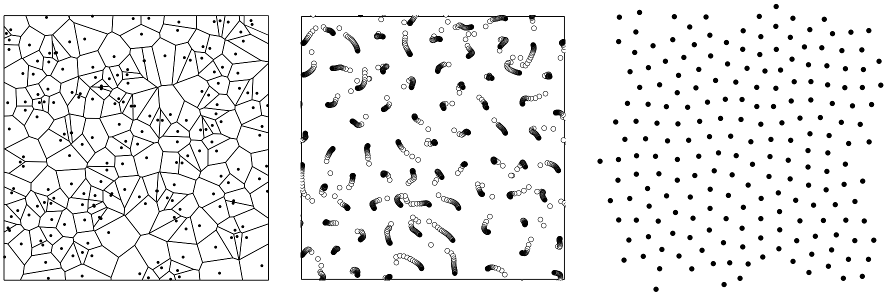

a) b) c)

Figure 1: Illustration of Lloyd's relaxation scheme: a) initial random point set with Voronoi Diagram, b) movement of points during iteration, and c) resulting point set after 20 iterations with periodic boundary conditions.

and minimizes the difference between maximal and minimal point-to-point distance. Thus blue noise spectral characteristics are obtained.

1.2 Analysis of Point Sets

Following the work of McCool and Fiume [8] the spectral properties of a two dimensional sampling point set $P_N := \{x_0, \ldots, x_{N-1}\} \subset [0,1)^2$ are analyzed by Fourier tools. Using the spectral power

$$R_f(\vec{\omega}) = \left| \mathcal{F} \frac{1}{N} \sum_{k=0}^{N-1} \delta(x - x_k) \right|^2$$

in the frequency domain which is found by the Fourier transform $\mathcal{F}$ of the point set P_N, we are able to graph the mean radial power

$$P_i = \frac{1}{S_i} \int_0^{2\pi} \int_{f_i}^{f_{i+1}} R_f(f\cos\theta, f\sin\theta) f df d\theta$$

and anisotropy

$$A_i = \frac{s_i^2}{P_i^2},$$

which allows to determine whether the point set has similar statistic characteristics in different directions. Here

$$\begin{aligned} s_i^2 &= \frac{1}{S_i} \int_0^{2\pi} \int_{f_i}^{f_{i+1}} (R_f(f\cos\theta, f\sin\theta) \\ &\qquad -P_i)^2 f df d\theta \end{aligned}$$

is the variance within annullus i. The annulli are defined by the frequency interval $[f_i, f_{i+1})$, where $S_i = \pi\left(f_{i+1}^2 - f_i^2\right)$ is the area of annullus ring i.

For a more complete analysis additionally the minimal and maximal distance between nearest neighbors are determined. This allows to detect even small irregularities.

2 Generating Blue Noise Samples

Instead of applying Lloyd's relaxation method as a post process for optimizing samples obtained by dart-throwing, the relaxation scheme can be applied directly to transform random samples into samples with blue noise characteristics.

One relaxation step determines the Voronoi region corresponding to each sample point and then moves this point into the center of gravity of its Voronoi region.

Since Lloyd's scheme is a strictly descending method it is important to use a sufficiently complex point set as input and only a small number of iterations, so that the method does not find the global minimum, wich is in most cases a hexagonal grid.

In Figure 1 the relaxation process is illustrated: (a) shows the initial random point set with its Voronoi diagram, in (b) the movement of the point is depicted, and (c) shows the final point set after 20 iterations. The relaxation was performed on the torus, i.e. using periodic boundary conditions so that the final point set tiles seamlessly and preserves blue noise characteristics across tile boundaries. The periodic boundary conditions are realized by wrapping around the points during the computation of the Voronoi regions.

Figure 2 shows the spectral properties of 200 points during the course of relaxation. Low fre-

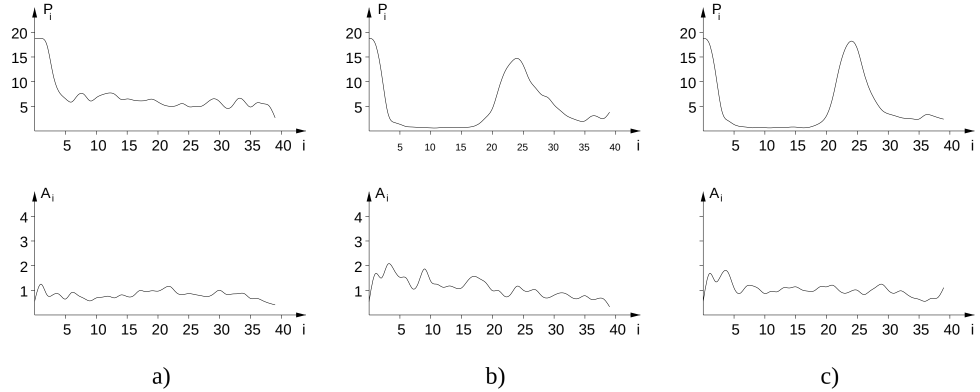

Figure 2: Development of mean radial power P_i and anisotropy A_i of the a) initial random point set, after b) 20 iteration steps of Lloyd's method, and after c) 80 steps.

quencies become attenuated and the minimal point-to-point distance is increased (also see the table below).

In Figure 3 the results of (a) dart throwing, and (b) the relaxation scheme applied to a random initial points are compared. As a result, lower frequencies are attenuated more effectively by the relaxation method, and medium frequencies are stronger. Both random point sets approximate uniform distribution. However, it becomes obvious that the price for the higher uniformity of the relaxed points is a slightly increased anisotropy. The range Δ of the nearest neighbor distances in Figure 3 is much smaller for the relaxation method (see the table below). This is mainly due to the increased minimum distance.

NN distance	max	min	Δ
Dart throwing	33.44	21.64	11.8
Lloyd's method	32.33	24.86	7.47

The computing time for both methods is similar: 80 relaxation steps for 1000 points are computed in about 40 seconds, the refined dart throwing method by McCool and Fiume needs 50 seconds, if a maximum number of 150.000 trials is allowed before the disk radius is decreased. If graphics hardware is used for computing the Voronoi diagram during the relaxation [6], on a standard PC 25 seconds are needed for 80 iterations with 1000 points.

2.1 Hierarchical Blue Noise Samples

For efficient adaptive sampling progressive point sets are required. While McCool and Fiume [8] realized this by decreasing the disk radius and re-running the dart throwing algorithm, also Lloyd's method can be used to add new points. This is done by using the relaxation to move the new points around the old points, which have been fixed before the iteration.

In Figure 4 the point sets resulting from adding (a) N, (b) $2N$, and (c) $3N$ points to a set of N blue noise samples are plotted, where the new points are marked by crosses. Obviously the attenuation of low frequencies by adding N new points is not that effective. In the two other cases, however, the low frequencies are effectively attenuated at the cost of a slightly increased anisotropy.

3 Aperiodic Tiling

The idea of tiling the sampling plane has been already mentioned by Dippé and Wold [3]. However they proposed to replicate rotations of one single pattern. By doing so blue noise characteristics will be destroyed across tile boundaries.

Instead more than one point set can be computed to aperiodically tile the plane. Therefore we developed a version of the Lloyd's method that synchronizes different point sets by relaxing one point set while multiple edge constraints are imposed by all the point sets that potentially can be placed at this

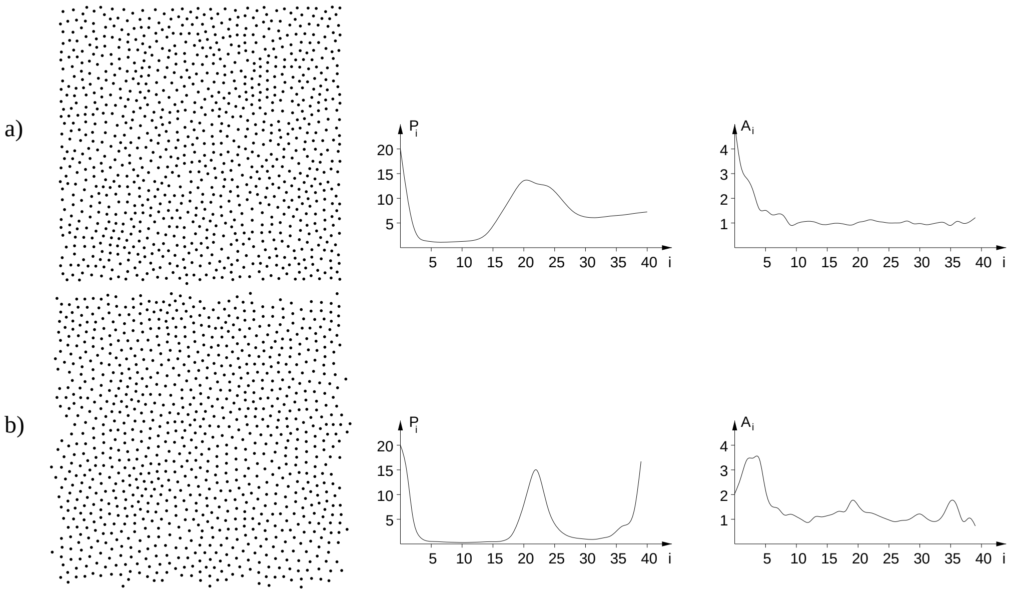

Figure 3: $N = 1000$ sample points with mean radial power P_i and anisotropy A_i generated by a) dart throwing and b) Lloyd's method.

edge.

In the original version of Lloyd's method, the point is moved into the center of gravity of its Voronoi region. This is done by calculating a displacement vector for each point. In our new tiling version of the algorithm the displacement vector is determined by the average of all displacement vectors for the Voronoi regions that occur if different point sets are placed side by side. This affects mostly the points at the border of each quadratic tile where for each neighbor different Voronoi regions are generated. This process of averaging is repeated for each point set until all of them fit together.

Figure 5 shows the process. Two different Voronoi diagrams are created by two different neighbors to a quadratic tile. Due to the different Voronoi regions a set of displacement vectors is produced that is averaged.

Generally, for a set of n point sets on quadratic tiles, n edge constraints are imposed on each edge of a tile. This number can be reduced by the use of Wang tiles [13], that allow for an aperiodic tiling of the plane by only 13 different tiles. Shade et al. [11] use an algorithm that works with eight tiles and produces no visible patterns, though its aperiodicity was not proven. In the latter case, each edge in each tile has only four possible neighbors. This allows the above algorithm produce results in reasonable time by dramatically reducing the effort required for synchronization.

Figure 6 shows the results for Wang-tiling point sets with (a) $N = 52$ points and (b) $N = 13$ points. Both distributions show convenient spectral properties that are similar to the results of untiled point sets that, however, are much more expensive to generate. In both cases slightly increased anisotropy values are observed, these are caused by the tiling process itself which introduces a grid structure in the Fourier transform though it is aperiodic (see Figure 7).

The overall computation time for the point sets with 13 points and 25 iterations with all possible neighbors is about 90 seconds on a Pentium III PC with 800 MHz, for the sets with 52 points and 25 iterations we need approximately 270 seconds.

4 Conclusion

We introduced an efficient method for generating sampling patterns with blue noise spectral properties. Using only a small number of progressive basis

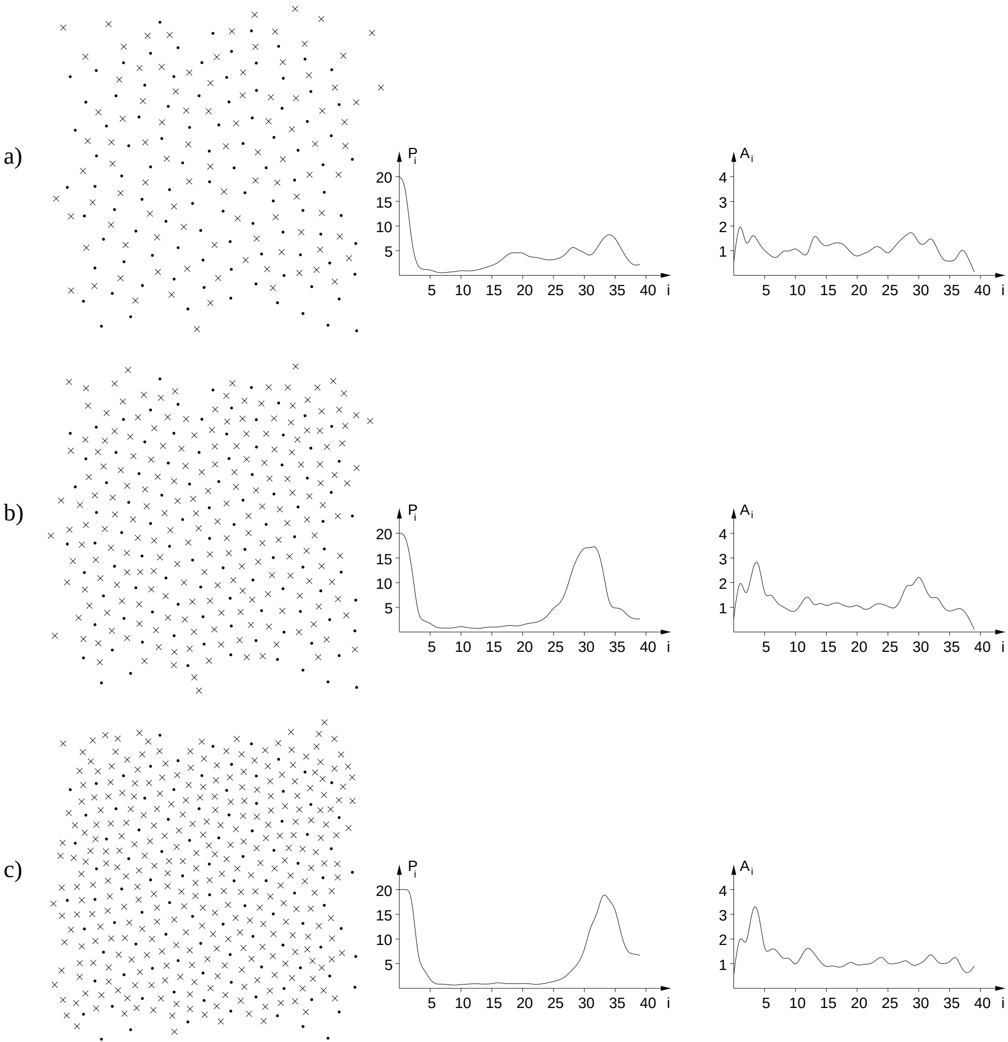

Figure 4: Hierarchical blue noise samples with with mean radial power P_i and anisotropy A_i: a) adding 100 points to a set of 100 points, b) adding 200 points, and c) adding 300 points. The added points are depicted by crosses.

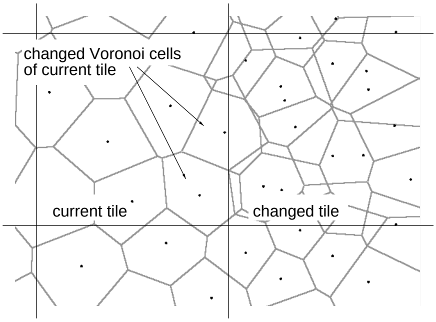

Figure 5: Relaxation with multiple edge constraints. The points in the *current tile* are moved by the movements determined by applying one iteration step for the boundary conditions imposed by the changed tile.

point sets, the plane is efficiently tiled in an aperiodic way, while still allowing for local hierarchical supersampling. The generated patterns are superior to Poisson disk distributions since by construction minimal and maximal point-to-point distances remain restricted.

Apart from efficient antialiasing, the point patterns can be used in several applications where point distributions with minimum distance properties are needed. Examples include the positioning of plants [1] within plant populations and stippling, which is a non-realistic rendering method using dots [2].

An interesting open question for future work concerns the relation of Lloyd's relaxation process to the adaptive sampling strategy of Eldar et al. [4].

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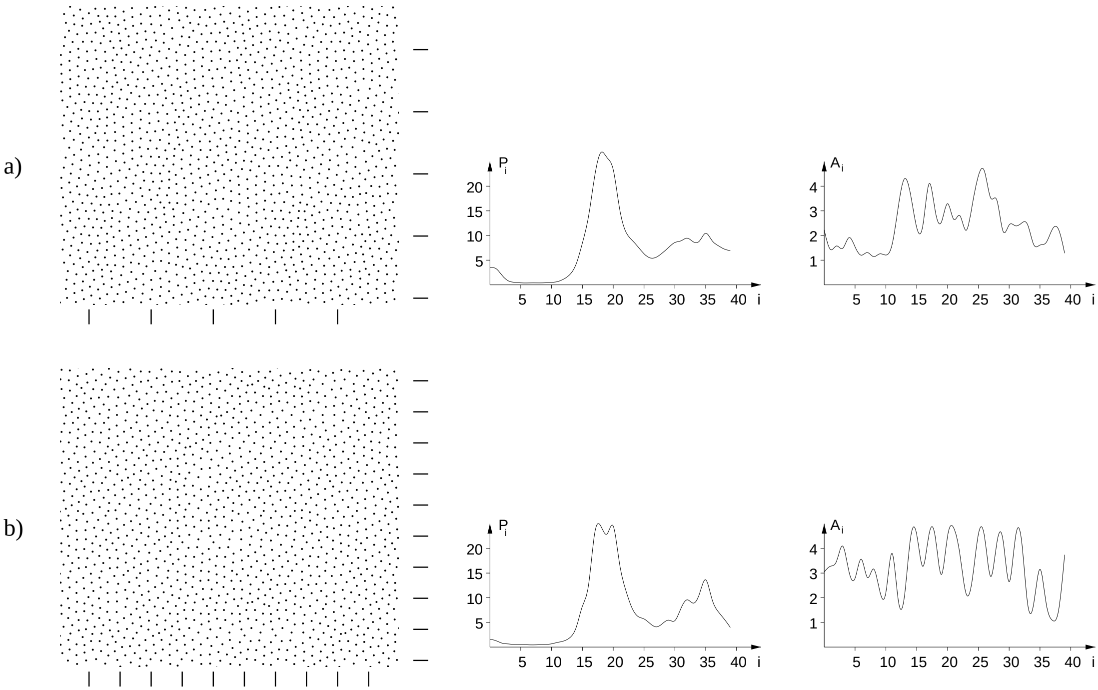
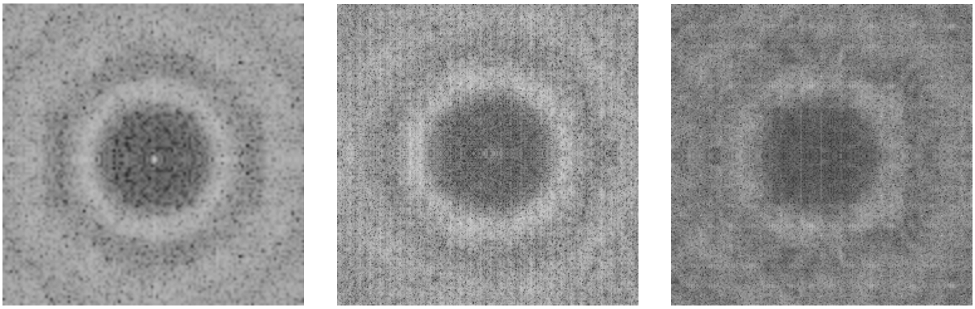


Figure 6: Aperiodically tiled blue noise samples with mean radial power P_i and anisotropy A_i using eight Wang tiles at a) 52 points per tile and b) 13 points per tile. The scratch marks besides the point sets indicate the tile boundaries.



a) b) c)

Figure 7: Fourier transform of a) a blue noise point set without tiling, b) aperiodically tiled point sets at 52 points per tile, and c) aperiodically tiled point sets at 13 points per tile. Weak grid structures are visible in the tiled instances. Using less points per tile increases the anisotropy.